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# Voting

# Lezione n. 12

# Corso di Laurea: Informatica

# Insegnamento: Sistemi multi-agente

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## **Reaching Agreements - Voting**

How do agents *reaching agreements* when they are self interested?

In an extreme case (zero sum encounter) no agreement is possible — but in most scenarios, there is potential for *mutually beneficial agreement* on matters of common interest.

The capabilities of *negotiation* are central to the ability of an agent to reach such agreements.

Negotiation is governed by a particular *mechanism*, or *protocol*

The mechanism defines the “rules of encounter” between agents

*Mechanism design* is designing mechanisms so that they have certain desirable properties

Given a particular protocol, how can a particular *strategy* be designed that individual agents can use?

Desirable properties of mechanisms:

*Convergence/guaranteed success*

*Maximizing social welfare*

*Pareto efficiency*

*Individual rationality (playing by the rules)*

*Stability (Nash equilibrium)*

*Simplicity*

*Distribution*



# Reaching Agreements – Voting

*(W: 7.1; MAS: 9.3.1, 9.3.2, 9.4.1, 9.4.2, 9.5)*

- does it make sense
  - to vote for a candidate you fancy least?
  - for a general, to toss a coin?
  - in poker, place a maximal bid with the worst cards?
  - to throw some goods away before starting to negotiate about them?
  - to sell your house to the second best bidder?

- You are baby sitting three children—Will, Liam, Vic—and need to decide on an activity for them.
- You can choose among going to the video arcade (a), playing basketball (b), and driving around in a car (c).
- Each kid has a different preference over these activities

Will:  $a > b > c$

Liam:  $b > c > a$

Vic:  $c > b > a$

- Let  $N = \{1, 2, \dots, n\}$  denote a set of agents.
- Let  $O$  denote a finite set of outcomes (or alternatives, or candidates).
- Let preference  $L$  be the set of strict total orders.

***A social choice function*** (over  $N$  and  $O$ ) is a function

$$C : L^n \rightarrow O.$$

If there exists a candidate  $x$  such that if for all other candidates  $y$  at least half the voters prefer  $x$  to  $y$ , then  $x$  must be chosen.

## Social Choice Example

- Given alternatives  $X = \{a, b\}$ , and the following preferences (each column is a preference order  $L$ , the first row indicates the number of players with that preference order):

3	7
<hr/>	
$a$	$b$
$b$	$a$

- Question: Which alternative (a or b) is preferred?
- Question: Formulate a social choice function  $f$

## Majority Voting with Two Alternatives

- Two alternatives  $a$  and  $b$ , three possible preference relations:

$$a > b \quad a \sim b \quad b > a$$

- Majority voting: Order the two candidates proportional to the number of “votes” they obtain.
- Social choice function  $f$  selects the candidate with the most votes.

## Majority Rule on More than Two Alternatives

- Question: Who should be the winner according to the majority rule?

3	5	7	6
<i>a</i>	<i>a</i>	<i>b</i>	<i>c</i>
<i>b</i>	<i>c</i>	<i>d</i>	<i>b</i>
<i>c</i>	<i>b</i>	<i>c</i>	<i>d</i>
<i>d</i>	<i>d</i>	<i>a</i>	<i>a</i>

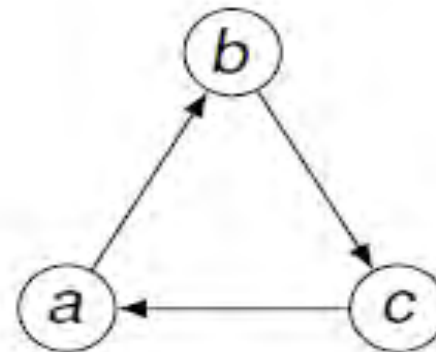
- If there exists a candidate  $x$  such that if for all other candidates  $y$  at least half the voters prefer  $x$  to  $y$ , then  $x$  must be chosen.

3	5	7	6
<i>a</i>	<i>a</i>	<i>b</i>	<i>c</i>
<i>b</i>	<i>c</i>	<i>d</i>	<i>b</i>
<i>c</i>	<i>b</i>	<i>c</i>	<i>d</i>
<i>d</i>	<i>d</i>	<i>a</i>	<i>a</i>

- Question: Who is the Condorcet winner?

- The Condorcet Paradox: A Condorcet winner does not always exist.

$\theta_1$	$\theta_2$	$\theta_3$
$a$	$c$	$b$
$b$	$a$	$c$
$c$	$b$	$a$



**Plurality voting:** *Each voter casts a single vote. The candidate with the most votes is selected.*

- tie-breaking rule

**Cumulative voting:** *Each voter is given  $k$  votes, which can be cast arbitrarily. The candidate with the most votes is selected.*

**Approval voting:** *Each voter can cast a single vote for as many of the candidates as he wishes; the candidate with the most votes is selected.*

### **Plurality with elimination:**

- *Each voter casts a single vote for their most-preferred candidate.*
- *The candidate with the fewest votes is eliminated.*
- *Each voter who cast a vote for the eliminated candidate casts a new vote for the candidate he most prefers among the candidates that have not been eliminated.*
- *This process is repeated until only one candidate remains.*

**Borda voting:**

- *Each voter submits a full ordering on the candidates.*
- *This ordering contributes points to each candidate; if there are  $n$  candidates, it contributes  $n-1$  points to the highest ranked candidate,  $n-2$  points to the second highest, and so on;*
- *It contributes no points to the lowest ranked candidate.*
- *The winners are those whose total sum of points from all the voters is maximal.*

## Borda cannot always select one winner

- **Example:**

$\theta_1$	$\theta_2$	$\theta_3$
$a$	$b$	$c$
$b$	$c$	$a$
$c$	$a$	$b$

- Question: Who is the Borda winner?

### **Pairwise elimination:**

- *In advance, voters are given a schedule for the order in which pairs of candidates will be compared.*
- *Given two candidates (and based on each voter's preference ordering) determine the candidate that each voter prefers.*
- *The candidate who is preferred by a minority of voters is eliminated, and the next pair of noneliminated candidates in the schedule is considered.*
- *Continue until only one candidate remains.*

Condorcet condition?

499 agents:  $a > b > c$

3 agents:  $b > c > a$

498 agents:  $c > b > a$

Plurality?

Plurality with elimination?

Borda?

Condorcet condition  $\rightarrow b$

499 agents:  $a > b > c$

3 agents:  $b > c > a$

498 agents:  $c > b > a$

Plurality  $\rightarrow a$

Plurality with elimination  $\rightarrow c$

Borda  $\rightarrow b$

35 agents:  $a \succ c \succ b$

33 agents:  $b \succ a \succ c$

32 agents:  $c \succ b \succ a$

Plurality?

Borda?

35 agents:  $a > c > b$

33 agents:  $b > a > c$

32 agents:  $c > b > a$

Plurality  $\rightarrow a$

Borda  $\rightarrow a$  (103, 98, 99)

What if  $c$  does not exist?

Plurality  $\rightarrow ?$

Borda  $\rightarrow ?$

35 agents:  $a > c > b$

33 agents:  $b > a > c$

32 agents:  $c > b > a$

Plurality  $\rightarrow a$

Borda  $\rightarrow a$  (103, 98, 99)

What if  $c$  does not exist?

Plurality  $\rightarrow b$

Borda  $\rightarrow b$

3 agents:  $a \succ b \succ c \succ d$

2 agents:  $b \succ c \succ d \succ a$

2 agents:  $c \succ d \succ a \succ b$

Borda method?

3 agents:  $a \succ b \succ c \succ d$

2 agents:  $b \succ c \succ d \succ a$

2 agents:  $c \succ d \succ a \succ b$

Borda method ranks the candidates  $c \succ b \succ a \succ d$ , with scores of 13, 12, 11, and 6.

Drop the lowest-ranked candidate  $d$

Borda?

3 agents:  $a \succ b \succ c \succ d$

2 agents:  $b \succ c \succ d \succ a$

2 agents:  $c \succ d \succ a \succ b$

Borda method ranks the candidates  $c \succ b \succ a \succ d$ , with scores of 13, 12, 11, and 6.

Drop the lowest-ranked candidate  $d$

Borda ranking is  $a \succ b \succ c$  with scores of 8, 7, and 6.

### Pairwise elimination method

35 agents:  $a > c > b$

33 agents:  $b > a > c$

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consider the order  $a, b, c$

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35 agents:  $a > c > b$

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consider the order  $a, b, c \rightarrow c$

consider the order  $a, c, b$

### Pairwise elimination method

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### Pairwise elimination method

35 agents:  $a \succ c \succ b$

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consider the order  $a, b, c \rightarrow c$

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consider the order  $b, c, a \rightarrow a$

The agenda setter can select whichever outcome he wants by selecting the appropriate elimination order!

Next, consider the following preferences.

1 agent:  $b > d > c > a$

1 agent:  $a > b > d > c$

1 agent:  $c > a > b > d$

Consider the elimination ordering  $a, b, c, d$

Next, consider the following preferences.

1 agent:  $b > d > c > a$

1 agent:  $a > b > d > c$

1 agent:  $c > a > b > d$

Consider the elimination ordering  $a, b, c, d \rightarrow d$  as the winner.

*However, all of the agents prefer  $b$  to  $d$ —the selected candidate is Pareto dominated!*

- Let  $N = \{1, 2, \dots, n\}$  denote a set of agents.
- Let  $O$  denote a finite set of outcomes (or alternatives, or candidates).
- Let preference  $L$  be the set of strict total orders.

*A **social welfare** function (over  $N$  and  $O$ ) is a function*

$$W : L^n \rightarrow L$$

- **Intuition:**
- **Anonymity:** The names of the players do not matter: if two players exchange types, the outcome is not affected.
- **Neutrality:** The names of the alternatives do not matter: if we exchange a and b in the preference profile of each agent, then the outcome is affected accordingly.

- **Definitions: Be  $f$  a social welfare function,**  
 $\mathbf{x, y \in X, (L1, \dots, Ln) \in L}$
- $f$  is anonymous if for every permutation  $\pi$  of  $L$ :  $f(L1, \dots, Ln) = f(L(\pi1), \dots, L(\pi n))$
- $f$  is neutral if for every permutation of  $X$ :  
 $f(\pi(L1), \dots, \pi(Ln)) = \pi(f(L1, \dots, Ln))$   
 (where  $a >_{\pi(Li)} b$  iff  $\pi(a) >_{Li} \pi(b)$ , for all  $a, b \in X$ )

## A Trivial Impossibility Result

- **Proposition: There is no anonymous and neutral social choice function.**
- Proof.
- Assume scf is anonymous and neutral. Consider ,  $L$ ,  $L'$ ,  $L''$ :

$\theta_1$	$\theta_2$	$\theta_3$	$\theta'_1$	$\theta'_2$	$\theta'_3$	$\theta''_1$	$\theta''_2$	$\theta''_3$
$a$	$b$	$c$	$b$	$c$	$a$	$a$	$b$	$c$
$b$	$c$	$a$	$c$	$a$	$b$	$b$	$c$	$a$
$c$	$a$	$b$	$a$	$b$	$c$	$c$	$a$	$b$

- W.l.o.g.,  $f(L) = a$ . For  $\pi(a) = b$ ,  $\pi(b) = c$ , and  $\pi(c) = a$ ,  $L' = \pi(L)$ .
- With neutrality,  $f(L') = \pi(f(L)) = \pi(a) = b$ .
- With anonymity,  $f(L') = f(L'') = b$ .
- However  $L = L''$ , a contradiction, since  $f(L) = a$ .

- **Intuition:**
- Pareto optimality: If alternative  $a$  is unanimously preferred to alternative  $b$ ,  $b$  should not be elected.
- Non-dictatorship: There is no player whose preference profile determines the strict preferences of the social welfare function.
- Unrestricted Domain: The social welfare function should define a social preference order for any given set of preference profiles.

$W$  is **Pareto efficient** if

for any  $o_1, o_2 \in O$ ,

$\forall i \ o_1 \succ_i o_2$  implies that  $o_1 \succ_W o_2$ .

when all agents agree on the ordering of two outcomes, the social welfare function must select that ordering.

$W$  is **independent of irrelevant alternatives** if, for any  $o_1, o_2 \in O$  and any two preference profiles  $[>'], [>''] \in L^n$ ,  $\forall i (o_1 >'_i o_2 \text{ if and only if } o_1 >''_i o_2)$  implies that  $(o_1 >_{W([>'])} o_2 \text{ if and only if } o_1 >_{W([>''])} o_2)$ .

the selected ordering between two outcomes should depend only on the relative orderings they are given by the agents.

## Nondictatorship

*W does not have a dictator if*


$$\neg \exists i \forall o_1, o_2 (o_1 \succ_i o_2 \Rightarrow o_1 \succ_W o_2).$$

there does not exist a single agent whose preferences always determine the social ordering.

We say that  $W$  is dictatorial if it fails to satisfy this property.

*If  $|O| \geq 3$ , any social welfare function  $W$  that is Pareto efficient and independent of irrelevant alternatives is dictatorial.*





Arrow's theorem tells us that we cannot hope to find a voting scheme that satisfies all of the notions of fairness that we find desirable.

Maybe the problem is that Arrow's theorem considers the identification of a social ordering over *all outcomes*.

Idea: social choice functions might be easier to find

We'll need to redefine our criteria for the social choice function setting; PE and IIA discussed the ordering

## Weak Pareto efficiency

*A social choice function  $C$  is weakly Pareto efficient if, for any preference profile  $[>] \in L^n$ , if there exist a pair of outcomes  $o_1$  and  $o_2$  such that  $\forall i \in N, o_1 \succ_i o_2$ , then  $C([>]) \neq o_2$ .*

A dominated outcome can't be chosen.

## Monotonicity

*C is monotonic if, for any  $o \in O$  and any preference profile  $[>] \in L^n$  with  $C([>]) = o$ , then for any other preference profile  $[>']$  with the property that*

$$\forall i \in N, \forall o' \in O, o >'_i o'$$

*if  $o >_i o'$ , it must be that  $C([>']) = o$ .*

an outcome  $o$  must remain the winner whenever the support for it is increased relative to a preference profile under which  $o$  was already winning

n.b. no constraint on the relative of outcomes  $o_1$  e  $o_2 \neq o$  (their relative order can be different)

## **Nondictatorship**

*C is nondictatorial if there does not exist an agent  $j$  such that  $C$  always selects the top choice in  $j$ 's preference ordering.*

*If  $|O| \geq 3$ , any social choice function  $C$  that is weakly Pareto efficient and monotonic is dictatorial.*

- Perhaps contrary to intuition, social choice functions are no simpler than social welfare functions after all.
- The proof repeatedly “probes” a social choice function to determine the relative social ordering between given pairs of outcomes.
- Because the function must be defined for all inputs, we can use this technique to construct a full social welfare ordering.

Plurality satisfies weak PE and ND, so it must not be monotonic.

Consider the following preferences:

3 agents:  $a > b > c$

2 agents:  $b > c > a$

2 agents:  $c > b > a$

Plurality chooses a.

Increase support for a by moving c to the bottom:

3 agents:  $a > b > c$

2 agents:  $b > c > a$

2 agents:  $b > a > c$

Now plurality chooses b.