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# Non-cooperative game theory (2)

**Lezione n.** 10

**Corso di Laurea:**
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# Distributed Rational Decision Making – Game Theory (2)

*(W: 3, MAS: 3.2.4, 3.3.3, 3.4.1, 5.1.2, 5.1.3)*

It would be a pretty bad idea to play any deterministic strategy in matching pennies

| | H | T |
|---|---|---|
| H | 1,-1 | -1,1 |
| T | -1,1 | 1,-1 |

Idea: confuse the opponent by playing randomly

Define a strategy si for agent i as any probability distribution over the actions Ai.

pure strategy: only one action is played with positive probability

mixed strategy: more than one action is played with positive probability

these actions are called the support of the mixed strategy

Let the set of all strategies for i be Si

Let the set of all strategy profiles be S = S1 x … x Sn.

## Utility under Mixed Strategies

What is your payoff if all the players follow mixed strategy profile s ∈ S?

We can't just read this number from the game matrix anymore: we won't always end up in the same cell

Instead, use the idea of expected utility from decision theory:

$$u_i(s) = \sum_{a \in A} u_i(a) Pr(a|s)$$

$$Pr(a|s) = \prod_{j \in N} s_j(a_j)$$

si(ai) we denote the probability that an action ai will be played under mixed strategy si.

**Every finite game has a Nash equilibrium! [Nash, 1950]**

e.g., matching pennies: both players play heads/tails 50%/50%

- It's hard in general to compute Nash equilibria, but it's easy when you can guess the support
- For BoS, let's look for an equilibrium where all actions are part of the support

| | B | F |
|---|---|---|
| B | 2, 1 | 0, 0 |
| F | 0, 0 | 1, 2 |

- Let player 2 play B with p, F with 1-p.
- If player 1 best-responds with a mixed strategy, player 2 must make him indifferent between F and B (why?)

- Let player 2 play B with p, F with 1-p.
- If player 1 best-responds with a mixed strategy, player 2 must make him indifferent between F and B (why?)

| | B | F |
|---|---|---|
| B | 2, 1 | 0, 0 |
| F | 0, 0 | 1, 2 |

$$u1(B)=u1(F)$$

$$2p+0(1-p)=0p+1(1-p)$$

$$p=1/3$$

## Computing Mixed Nash Equilibria: Battle of the Sexes

- Likewise, player 1 must randomize to make player 2 indifferent.
- Why is player 1 willing to randomize?
- Let player 1 play B with q, F with (1-q)

| | B | F |
|---|---|---|
| B | 2, 1 | 0, 0 |
| F | 0, 0 | 1, 2 |

$$u2(B)=u2(F)$$

$$q+0(1-q)=0q+2(1-q)$$

$$q=2/3$$

Thus the mixed strategies (2/3, 1/3), (1/3, 2/3) are a Nash equilibrium.

What does it mean to play a mixed strategy? Different interpretations:

- Randomize to confuse your opponent
  - consider the matching pennies example
- Players randomize when they are uncertain about the other's action
  - consider battle of the sexes

- Mixed strategies are a concise description of what might happen in repeated play: count of pure strategies in the limit
- Mixed strategies describe population dynamics: 2 agents chosen from a population, all having deterministic strategies. MS is the probability of getting an agent who will play one PS or another.

How hard is it to compute the Nash equilibria of a game?

Two-player, zero-sum games:

The Nash equilibrium problem for such games can be expressed as a *linear program (LP), which* means that equilibria can be computed in polynomial time.

Nash equilibrium of a two-player, general sum game, cannot be formulated as a linear program.

Essentially, this is because the two players' interests are no longer diametrically opposed.

NP-complete

*The following problems are NP-hard when applied to Nash equilibria:* uniqueness*, Pareto optimality, guaranteed social welfare.*

*Computing all of the equilibria of a two-player, general-sum game requires worst-case time that is exponential in the number of actions for each player.*

- The maxmin strategy of player i in an n-player, general-sum game is a strategy that maximizes i’s worst-case payoff
  - in the situation where all the other players happen to play the strategies which cause the greatest harm to i.

- The maxmin value (or security level) of the game for player i is that minimum amount of payoff guaranteed by a maxmin strategy.

Let $G = <\{1,2\},(Ai),(\geq i)>$ a zero sum-game

Action $x^* \in A1$ is maxminimizer for 1:

$\forall x \in A1 \quad \min u_1(x^*,y) > \min u_1(x,y)$

Action $y^* \in A2$ is maxminimizer for 2:

$\forall y \in A2 \quad \min u_2(x,y^*) > \min u_2(x,y)$

*The best case among the worsts*

Action $x^* \in A1$ is maxminimizer for 1:

$\forall x \in A1 \min u_1(x^*,y) \geq \min u_1(x,y)$

*maximises the minimum that I can guarantee*

$x^*$ is a security strategy for 1

Solves for 1 $\max_x \min_y u_1(x,y)$

Solves for 2 $\max_y \min_x u_2(x,y)$

$(x^*,y^*)$ is a N.eq for G, iff:

$x^*$ is a maxminimizer for 1;

$y^*$ is a maxminimizer for 2

$\max_x \min_y u_1(x,y)$
$\max_y \min_x u_2(x,y)$

$u_1(x^*,y^*)$

Solves for 1

$\max_x \min_y u_1(x,y) =$

$\max\{$

$\min\{u_1(x,y) | y \in A_2\}$

$| x \in A_1\}$

| | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
|---|---|---|---|---|---|
| $x_1$ | 2,-2 | 2,-2 | 3,-3 | 1,-1 | 1,-1 |
| $x_2$ | 3,-3 | 5,-5 | 4,-4 | 6,-6 | 4,-4 |
| $x_3$ | 5,-5 | 2,-2 | 4,-4 | 3,-3 | 3,-3 |
| $x_4$ | 6,-6 | 8,-8 | 5,-5 | 7,-7 | 5,-5 |
| $x_5$ | 3,-3 | 5,-5 | 4,-4 | 2,-2 | 3,-3 |
| $x_6$ | 4,-4 | 3,-3 | 6,-6 | 5,-5 | 4,-4 |

Solves for 1

$\max_x \min_y u_1(x,y) =$

$\max\{$

$\min\{u_1(x,y) | y \in A_2\}$

$| x \in A_1\}$

$x_1 = \min_y u_1(x_1, y) = 1$

| | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
|---|---|---|---|---|---|
| $x_1$ | 2,-2 | 2,-2 | 3,-3 | 1,-1 | 1,-1 |
| $x_2$ | 3,-3 | 5,-5 | 4,-4 | 6,-6 | 4,-4 |
| $x_3$ | 5,-5 | 2,-2 | 4,-4 | 3,-3 | 3,-3 |
| $x_4$ | 6,-6 | 8,-8 | 5,-5 | 7,-7 | 5,-5 |
| $x_5$ | 3,-3 | 5,-5 | 4,-4 | 2,-2 | 3,-3 |
| $x_6$ | 4,-4 | 3,-3 | 6,-6 | 5,-5 | 4,-4 |

Solves for 1

$\max_x \min_y u_1(x,y) =$

$\max\{$

$\min\{u_1(x,y) | y \in A_2\}$

$| x \in A_1\}$

$x_1 = \min_y u_1(x_1,y) = 1$

$x_2 = \min_y u_1(x_2,y) = 3$

…..

$x_6 = \min_y u_1(x_6,y) = 3$

| | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
|---|---|---|---|---|---|
| $x_1$ | 2,-2 | 2,-2 | 3,-3 | 1,-1 | 1,-1 |
| $x_2$ | 3,-3 | 5,-5 | 4,-4 | 6,-6 | 4,-4 |
| $x_3$ | 5,-5 | 2,-2 | 4,-4 | 3,-3 | 3,-3 |
| $x_4$ | 6,-6 | 8,-8 | 5,-5 | 7,-7 | 5,-5 |
| $x_5$ | 3,-3 | 5,-5 | 4,-4 | 2,-2 | 3,-3 |
| $x_6$ | 4,-4 | 3,-3 | 6,-6 | 5,-5 | 4,-4 |

Solves for 1

$\max_x \min_y u_1(x,y) =$

$\max\{$

$\min\{u_1(x,y) | y \in A_2\}$

$| x \in A_1\}$

$x_1 = \min_y u_1(x_1,y) = 1$

$x_2 = \min_y u_1(x_2,y) = 3$

…..

$x_6 = \min_y u_1(x_6,y) = 3$

max = 5 for $x^* = x_4$

| | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
|---|---|---|---|---|---|
| $x_1$ | 2,-2 | 2,-2 | 3,-3 | 1,-1 | 1,-1 |
| $x_2$ | 3,-3 | 5,-5 | 4,-4 | 6,-6 | 4,-4 |
| $x_3$ | 5,-5 | 2,-2 | 4,-4 | 3,-3 | 3,-3 |
| $x_4$ | 6,-6 | 8,-8 | 5,-5 | 7,-7 | 5,-5 |
| $x_5$ | 3,-3 | 5,-5 | 4,-4 | 2,-2 | 3,-3 |
| $x_6$ | 4,-4 | 3,-3 | 6,-6 | 5,-5 | 4,-4 |

Solves for 2

$\max_y \min_x u_2(x,y) =$

$\max\{$

$\min\{u_2(x,y) | x \in A_1\}$

$| y \in A_2\}$

| | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
|---|---|---|---|---|---|
| $x_1$ | 2,-2 | 2,-2 | 3,-3 | 1,-1 | 1,-1 |
| $x_2$ | 3,-3 | 5,-5 | 4,-4 | 6,-6 | 4,-4 |
| $x_3$ | 5,-5 | 2,-2 | 4,-4 | 3,-3 | 3,-3 |
| $x_4$ | 6,-6 | 8,-8 | 5,-5 | 7,-7 | 5,-5 |
| $x_5$ | 3,-3 | 5,-5 | 4,-4 | 2,-2 | 3,-3 |
| $x_6$ | 4,-4 | 3,-3 | 6,-6 | 5,-5 | 4,-4 |

Solves for 2

$\max_x \min_y u_2(x,y)=$

$\max\{$

$\min\{u_1(x,y)|x\in A_1\}$

$|y\in A_2\}=-5$

For $y_5$

| | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
|---|---|---|---|---|---|
| $x_1$ | 2,-2 | 2,-2 | 3,-3 | 1,-1 | 1,-1 |
| $x_2$ | 3,-3 | 5,-5 | 4,-4 | 6,-6 | 4,-4 |
| $x_3$ | 5,-5 | 2,-2 | 4,-4 | 3,-3 | 3,-3 |
| $x_4$ | 6,-6 | 8,-8 | 5,-5 | 7,-7 | 5,-5 |
| $x_5$ | 3,-3 | 5,-5 | 4,-4 | 2,-2 | 3,-3 |
| $x_6$ | 4,-4 | 3,-3 | 6,-6 | 5,-5 | 4,-4 |

# Equilibrium (5,-5)

$(x_4,y_5)$

| | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
|---|---|---|---|---|---|
| $x_1$ | 2,-2 | 2,-2 | 3,-3 | 1,-1 | 1,-1 |
| $x_2$ | 3,-3 | 5,-5 | 4,-4 | 6,-6 | 4,-4 |
| $x_3$ | 5,-5 | 2,-2 | 4,-4 | 3,-3 | 3,-3 |
| $x_4$ | 6,-6 | 8,-8 | 5,-5 | 7,-7 | 5,-5 |
| $x_5$ | 3,-3 | 5,-5 | 4,-4 | 2,-2 | 3,-3 |
| $x_6$ | 4,-4 | 3,-3 | 6,-6 | 5,-5 | 4,-4 |

# Minmax Strategies

Player i's minmax strategy against player -i in a 2-player game is a strategy that minimizes -i's best-case payoff, and the minmax value for i against -i is payoff.

Why would i want to play a minmax strategy?

In a two-player game, the minmax strategy for player $i$ against player $-i$ is $\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$, and player $-i$'s minmax value is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$.

## Minimax theorem (von Neumann, 1928)

In any finite, two-player, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.

• Each player's maxmin value is equal to his minmax value. By convention, the maxmin value for player 1 is called the value of the game.

• For both players, the set of maxmin strategies coincides with the set of minmax strategies.

• Any maxmin strategy profile (or, equivalently, minmax strategy profile) is a Nash equilibrium.

•These are all the Nash equilibria.

# Game with Sequential Actions

Normal-form representation is universal

Extensive-form games (not simultaneous actions)

- Exponentially smaller than the normal-form.
- The normal-form game representation does not incorporate any notion of sequence, or time, of the actions of the players.
- The *extensive (or tree) form is an alternative* representation that makes the temporal structure explicit.

A pure strategy for a player in a perfect-information game is a complete specification of which deterministic action to take at every node belonging to that player.

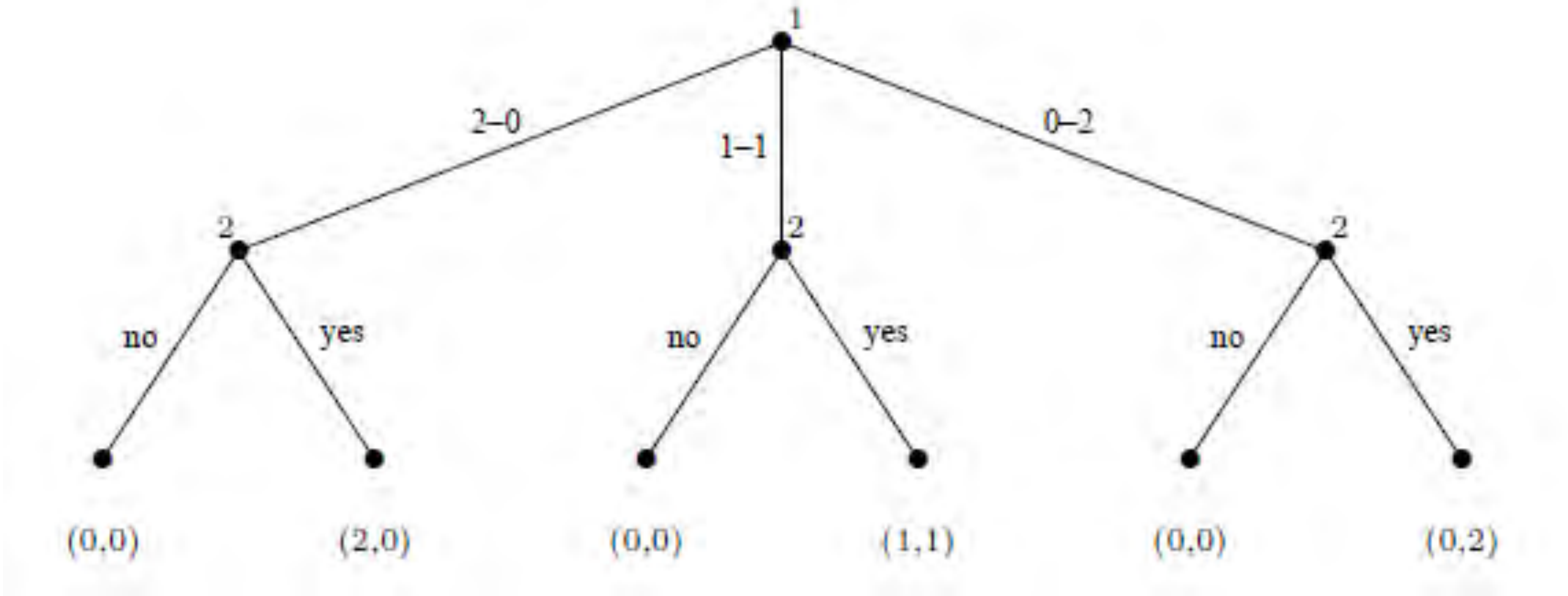
N.b. An agent's strategy requires a decision at each choice node, regardless of whether or not it is possible to reach that node given the other choice nodes.

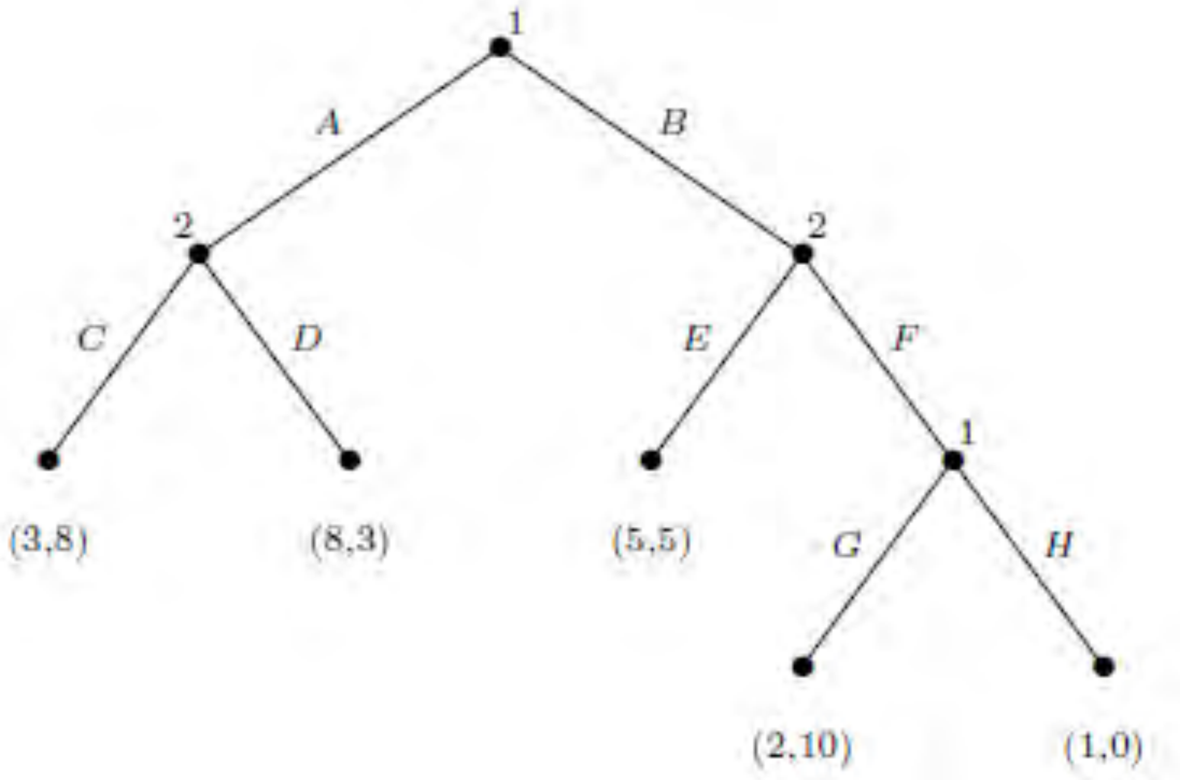
A brother and sister following the following protocol for sharing two indivisible and identical presents from their parents.

- First the brother suggests a split, which can be one of three—he keeps both, she keeps both, or they each keep one.
- Then the sister chooses whether to accept or reject the split.
- If she accepts they each get their allocated present(s), and otherwise neither gets any gift.

S1 = {2–0, 1–1, 0–2}

S2 = {(yes, yes, yes), (yes, yes, no), (yes, no, yes), (yes, no, no), (no, yes, yes), (no, yes, no), (no, no, yes), (no, no, no)}






S1 = {(A,G), (A,H), (B,G), (B,H)}

S2 = {(C,E), (C, F), (D,E), (D, F)}

| | (C,E) | (C,F) | (D,E) | (D,F) |
|---|---|---|---|---|
| (A,G) | 3,8 | 3,8 | 8,3 | 8,3 |
| (A,H) | 3,8 | 3,8 | 8,3 | 8,3 |
| (B,G) | 5,5 | 2,10 | 5,5 | 2,10 |
| (B,H) | 5,5 | 1,0 | 5,5 | 1,0 |

This transformation can always be performed

- it can result in an exponential blowup of the game representation.

The reverse transformation (from the normal form to the perfect-information extensive form) does not always exist.

***Every (finite) perfect-information game in extensive form has a pure-strategy Nash equilibrium.***

| | (C,E) | (C,F) | (D,E) | (D,F) |
|---|---|---|---|---|
| (A,G) | 3,8 | 3,8 | 8,3 | 8,3 |
| (A,H) | 3,8 | 3,8 | 8,3 | 8,3 |
| (B,G) | 5,5 | 2,10 | 5,5 | 2,10 |
| (B,H) | 5,5 | 1,0 | 5,5 | 1,0 |

| | (C,E) | (C,F) | (D,E) | (D,F) |
|---|---|---|---|---|
| (A,G) | 3,8 | 3,8 | 8,3 | 8,3 |
| (A,H) | 3,8 | 3,8 | 8,3 | 8,3 |
| (B,G) | 5,5 | 2,10 | 5,5 | 2,10 |
| (B,H) | 5,5 | 1,0 | 5,5 | 1,0 |

| | (C,E) | (C,F) | (D,E) | (D,F) |
|---|---|---|---|---|
| (A,G) | 3,8 | 3,8 | 8,3 | 8,3 |
| (A,H) | 3,8 | 3,8 | 8,3 | 8,3 |
| (B,G) | 5,5 | 2,10 | 5,5 | 2,10 |
| (B,H) | 5,5 | 1,0 | 5,5 | 1,0 |

| | (C,E) | (C,F) | (D,E) | (D,F) |
|---|---|---|---|---|
| (A,G) | 3,8 | 3,8 | 8,3 | 8,3 |
| (A,H) | 3,8 | 3,8 | 8,3 | 8,3 |
| (B,G) | 5,5 | 2,10 | 5,5 | 2,10 |
| (B,H) | 5,5 | 1,0 | 5,5 | 1,0 |

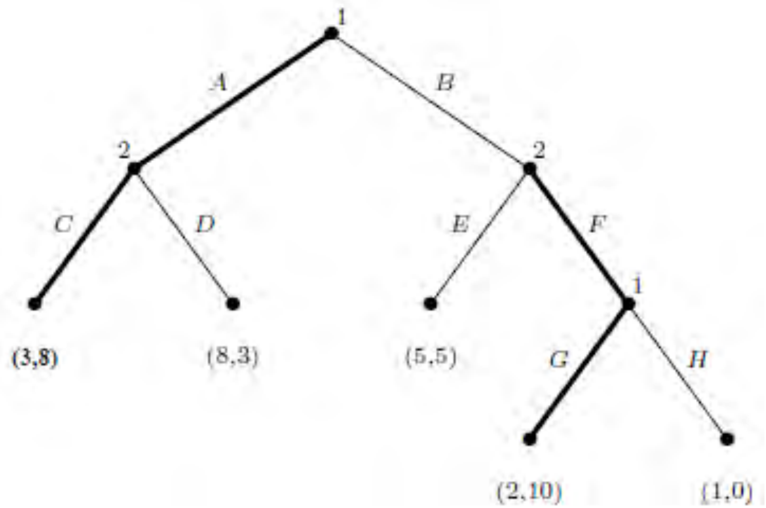
| | (C,E) | (C,F) | (D,E) | (D,F) |
|---|---|---|---|---|
| (A,G) | 3,8 | 3,8 | 8,3 | 8,3 |
| (A,H) | 3,8 | 3,8 | 8,3 | 8,3 |
| (B,G) | 5,5 | 2,10 | 5,5 | 2,10 |
| (B,H) | 5,5 | 1,0 | 5,5 | 1,0 |

Player 1 plays a threat (look at the tree)

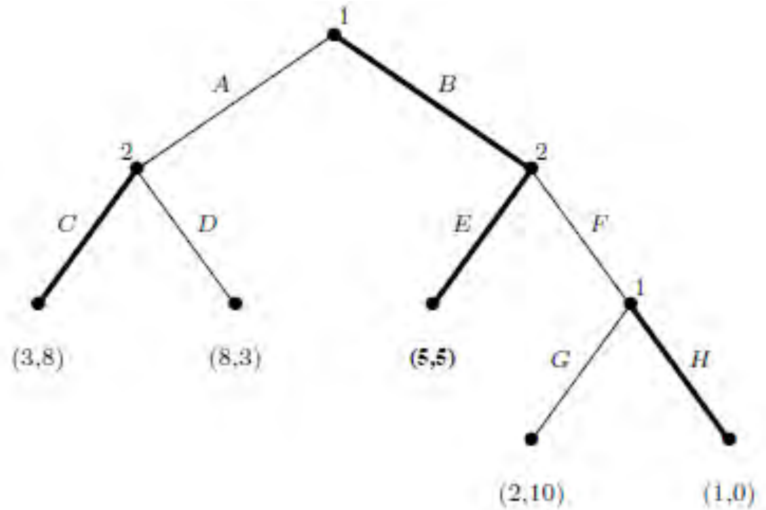
Is the threat believable?

Nash equilibrium can be too weak a notion for the extensive form.

{(A,G), (C, F)}




{(B,H), (C,E)}



*Given a perfect-information extensive-form game G, the subgame of G rooted at node h is the restriction of G to the descendants of h.*

*The set of subgames of G consists of all of subgames of G rooted at some node in G.*

*The subgame-perfect equilibria (SPE) of a game G are all strategy profiles s such that for any subgame G′ of G, the restriction of s to G′ is a Nash equilibrium of G′.*

SPE is also a Nash equilibrium

- every perfect-information extensive-form game has at least one subgame-perfect equilibrium.

not every NE is a SPE

| | (C,E) | (C,F) | (D,E) | (D,F) |
|---|---|---|---|---|
| (A,G) | 3,8 | 3,8 | 8,3 | 8,3 |
| (A,H) | 3,8 | 3,8 | 8,3 | 8,3 |
| (B,G) | 5,5 | 2,10 | 5,5 | 2,10 |
| (B,H) | 5,5 | 1,0 | 5,5 | 1,0 |

# Every strategy with H cannot be a SPE